1 Limits, Part 0 (Prereqs)

Before we formally define limits, it helps to have a handwavy intuition for limits mechanics, and to understand inequalities. In this chapter we learn these prereqs as quickly as possible.

1.1 Handwavy limits definition

Here I only present a hand-wavy definition of limits and use it to explain the mechanics of computing limits of functions in practice. A proper definition and proofs of the theorems that make the mechanics work come in a later chapter.

A hand-wavy definition: a limit of $f(x)$ at a is the value $f(x)$ approaches close to (but not necessarily at) a.

A slightly less hand-wavy definition: let $f : \mathbf{R} \to \mathbf{R}$, let $a \in \mathbf{R}$ be some number on the x-axis, and let $l \in \mathbf{R}$ be some number on the y-axis. Then as x gets closer to a, $f(x)$ gets closer to l.

The notation for this whole thing is

$$\lim_{x \to a} f(x) = l$$

So for example $\lim_{x \to 5} x^2 = 25$ because the closer x gets to 5, the closer x^2 gets to 25 (we'll prove all this properly soon). Now suppose you have some fancy pants function like this one:

$$\lim_{x \to 0} \frac{1 - \sqrt{x}}{1 - x} \tag{1}$$

If you plot it, it's easy to see that as x approaches 0, the whole shebang approaches 1. But how do you algebraically evaluate the limit of this thing? Can you just plug 0 into the equation? It seems to work, but once we formally define limits, we'll have to prove somehow that plugging $a = 0$ into x gives us the correct result.

1.2 Limits evaluation mechanics

It turns out that it does in fact work because of a few theorems that make practical evaluation of many limits easy. Here I'll state these theorems as facts. Once I introduce the formal definition of limits in a later chapter I'll properly prove them.

1. **Constants**. $\lim_{x \to a} c = c$, where $c \in \mathbf{R}$. In other words if the function is a constant, e.g. $f(x) = 5$, then $\lim_{x \to a} f(x) = 5$ for any a.
2. **Identity**. $\lim_{x \to a} x = a$. In other words if the function is an identity function $f(x) = x$, then $\lim_{x \to 6} f(x) = 6$. Meaning we simply plug a into x.

3. **Addition**[1]. $\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$. For example $\lim_{x\to a}(x+2) = \lim_{x\to a} x + \lim_{x\to a} 2 = a+2$.

4. **Multiplication**. $\lim_{x\to a}(f\cdot g)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$. For example $\lim_{x\to a} 2x = \lim_{x\to a} 2 \cdot \lim_{x\to a} x = 2a$.

5. **Reciprocal**. $\lim_{x\to a}\left(\frac{1}{f}\right)(x) = \frac{1}{\lim_{x\to a} f(x)}$ when the denominator isn't zero. For example $\lim_{x\to a}\frac{1}{x} = \frac{1}{\lim_{x\to a} x} = \frac{1}{a}$ for $a \neq 0$.

To come back to 1, these theorems tells us that

$$\lim_{x\to 0}\frac{1-\sqrt{x}}{1-x} = \frac{\lim_{x\to 0} 1 - (\lim_{x\to 0} x)^{\frac{1}{2}}}{\lim_{x\to 0} 1 - \lim_{x\to 0} x} = \frac{1-0^{\frac{1}{2}}}{1-0} = 1$$

Holes

What happens if we try to take a limit as $x \to 1$ rather than $x \to 0$?

$$\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x}$$

We can't use the same trick and plug in 1 because we get a nonsensical result 0/0 as the function isn't defined at 0. If we plot it, we clearly see the limit approaches 1/2 at 0, but how do we prove this algebraically? The answer is to do some trickery to find a way to cancel out the inconvenient term (in this case $1-\sqrt{x}$)

$$\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1}\frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})} = \lim_{x\to 1}\frac{1}{1+\sqrt{x}} = \frac{1}{2}$$

Why is it ok here to divide by $1-\sqrt{x}$? Good question! Recall that the limit is defined *close to* a (or *around* a, or as x *approaches* a), but not **at** a. In other words $f(a)$ need not even be defined (as is the case here). This means that as we consider $1-\sqrt{x}$ at different values of x as it approaches a, the limit never requires us to evaluate the function at $x = a$. So we never have to consider $1-\sqrt{x}$ as $x = 1$, $1-\sqrt{x}$ never takes on the value of 0, and it is safe to divide it out.

1.3 Absolute value inequalities

Consider an inequality $0 < |x-a| < \delta$. This will come up a lot soon. What does this inequality mean? The intuitive reading is that the difference between x and a is between 0 and δ. But it's a little subtle, so let's look at it carefully. There are actually two inequalities here: $0 < |x-a|$ and $|x-a| < \delta$. We should consider each separately.

[1]Spivak's book uses a slightly more verbose definition that assumes the limits of f and g exist near a, see p. 103

The left side, $0 < |x - a|$ is equivalent to $|x - a| > 0$. But $|x - a|$ is an absolute value, it's **always** true that $|x - a| \geq 0$. So this part of the inequality says $x - a \neq 0$, or $x \neq a$. I don't know why mathematicians say $0 < |x - a|$ instead of $x \neq a$, probably because confusing you brings them pleasure.

The right side is $|x - a| < \delta$. Intuitively this says that the difference between x and a should be less than δ. Put differently, x should be within δ of a. Algebraically we can write it as two cases:

1. $x - a < \delta$
2. $-(x - a) < \delta$

A little basic manipulation, and we can rewrite this as $a - \delta < x < a + \delta$.

1.4 Bounding with inequalities

We will often need to make an inequality of the following form work out:

$$|n||m| < \epsilon$$

Here ϵ is given to us, we have complete control over the upper bound of $|n|$, and $|m|$ can take on values outside our direct control. Obviously we can't make the inequality work without knowing *something* about $|m|$, so we'll try to find a bound for it in terms of other fixed values, or values we control.

For example, suppose we've discovered there is a fixed value a, and that $|m| < 3|a| + 4$. Given that we control $|n|$, how do we bound it in terms of ϵ and $|a|$ in such a way that the inequality $|n||m| < \epsilon$ holds?

Since we control $|n|$ and $(3|a| + 4)$ is fixed, we can find $|n|$ small enough so that $|n|(3|a| + 4) < \epsilon$ holds. Then certainly any inequality whose left side is smaller, e.g. $|n|(3|a| + 3) < \epsilon$, will also hold. And since $|m|$ is always smaller than $3|a| + 4$, it follows $|n||m| < \epsilon$ will hold as well.

All we have left to do is find a bound for $|n|$ such that $|n|(3|a| + 4) < \epsilon$ holds, which is of course easy:

$$|n| < \frac{\epsilon}{3|a| + 4}$$

Having bound $|n|$ in this way, we can verify that $|n|(3|a| + 4) < \epsilon$ holds by multiplying both sides of the above inequality by $3|a| + 4$.