5 Complete ordered fields

5.1 Motivation

The twelve ordered field axioms are sufficient to define limits, continuity, and prove all the theorems in the previous sections. Since the set $\mathcal{Q}$ of rational numbers is an ordered field[4], rationals have been sufficient for the work we've done so far. However, we are about to start proving slightly more sophisticated theorems about continous functions, and ordered fields will quickly start breaking our intuitions.

For example, consider the function $f(x) = x^2 - 2$ (a parabola shifted down two units). It's easy to see f is a continuous function, and thus our intuition is that we should be able to draw it without "lifting the tip of the pencil off the sheet of paper". Upon reflection however, it becomes obvious that in the universe limited to ordered fields this is impossible. f intersects the x-axis when $x^2 = 2$, but every high school student knows $\sqrt{2} \notin \mathcal{Q}$ (see 5.1 for proof). Thus there is no $x \in \mathcal{Q}$ such that $f(x) = 0$. And since $\mathcal{Q}$ is an ordered field, it follows ordered fields alone aren't sufficient to resolve this problem.

The *intermediate value theorem* (see 6.1) formalizes the claim that a continuous function segment that starts below the x-axis and ends above the x-axis intersects the x-axis. But as we can see from the example above, this is not possible to prove with ordered field axioms alone. So before we proceed with further study of continuity, we need one more axiom called *the completeness axiom*, which we introduce in this chapter.

Combined with the twelve ordered field axioms, the completeness axiom forms *complete ordered fields*. These objects are sufficient to proceed with our study of calculus. We will see that rational numbers $\mathcal{Q}$ are not a complete ordered field, whereas real numbers $\mathcal{R}$ are.[5] Thus from here $\mathcal{R}$-valued functions will become our primary object of study.

Aside: sqrt(2) is irrational

Suppose $\sqrt{2} \in \mathcal{Q}$. Then there exist $a, b \in \mathcal{N}$ such that $\left(\frac{a}{b}\right)^2 = 2$. Assume a, b have no common divisor (since we can obviously keep simplifying until this is the case). Observe that both a and b cannot be even, otherwise we could simplify further.

Now we have $a^2 = 2b^2$. Thus a^2 is even, a must be even[6], and there exists $k \in \mathcal{N}$ such that $a = 2k$. Then $a^2 = 4k^2 = 2b^2$ so $2k^2 = b^2$. Thus b^2 is even and

[4] The proof is straightforward, so I'm not including it here.

[5] Proof that $\mathcal{R}$ is a complete ordered field requires construction of $\mathcal{R}$, which doesn't happen in Spivak until the last chapters. Thus I will not be delving into that here and ask the reader (i.e., currently myself) to take this on faith.

[6] Even numbers have even squares because $(2k)^2 = 4k^2 = 2 \cdot (2k^2)$

so b is even. Since both a and b cannot be even, this is a contradiction. Thus $\sqrt{2} \notin \mathcal{Q}$ as desired.

5.2 Least Upper Bound

Definition: b is an **upper bound** for S if $s \leq b$ for all $s \in S$.

For example:

- Any $b \geq 1$ is an upper bound for $S = \{x : 0 \leq x < 1\}$. E.g. $1, 2, 10$ are all upper bounds of S.
- By convention, *every* number is an upper bound for $\emptyset$.
- The set $\mathcal{N}$ of natural numbers has no natural upper bound. The proof is easy. Suppose $b \in \mathcal{N}$ is an upper bound for $\mathcal{N}$. But $b + 1 \in \mathcal{N}$, and $b + 1 > b$, which is a contradiction. Thus b isn't an upper bound for $\mathcal{N}$.[7]

Definition: x is a **least upper bound** of A, if

1. x is an upper bound of A,
2. *and* if y is an upper bound of A, then $x \leq y$.

A set can have only one least upper bound. The proof is easy. Suppose x and x' are both least upper bounds of S. Then $x \leq x'$ and $x' \leq x$. Thus $x = x'$. Consequently, we can use a convenient notation $\sup A$ to denote the least upper bound of A.

Obligatory examples:

- Let $S = \{x : 0 \leq x < 1\}$. Then $\sup S = 1$.
- By convention, the empty set $\emptyset$ has no least upper bound.

5.3 Completeness axiom

We are now ready to state the completeness axiom.

Completeness [P13]: If A is a non-empty set of numbers that has an upper bound, then it has a least upper bound.

Claim: rational numbers are not complete.
Proof: Let $C = \{x : x^2 < 2 \text{ and } x \in \mathcal{Q}\}$. Suppose for contradiction rational numbers are complete. Then there exists $b \in \mathcal{Q}$ such that $b = \sup C$. Observe that

[7] We need to do a little more work to show $\mathcal{N}$ has no upper bound, natural or not. Be patient! We will prove this by the end of the section.

- $b^2 \neq 2$ as that would imply $b = \sqrt{2}$ and thus $b \notin \mathcal{Q}$.
- $b^2 \not< 2$ as there would exist some $x \in C$ such that $b^2 < x^2 < 2$. Thus $b < x$ and b is not the upper bound.

Therefore $b^2 > 2$. But this implies there exists some $x \in \mathcal{Q}$ such that $2 < x^2 < b^2$. Thus x is greater than every element in C, and $x < b$. So b is not the *least* upper bound. We have a contradiction, therefore rational numbers are not complete, as desired.

Claim: completeness cannot be derived from ordered fields.
Proof: $\mathcal{Q}$ is not complete and $\mathcal{Q}$ is an ordered field. Thus completeness is not a property of ordered fields.

Claim: real numbers are complete.
Proof [deferred]: The completeness property can be derived from the construction of real numbers $\mathcal{R}$, which makes reals a **complete ordered field**. The proof requires we study the actual construction of $\mathcal{R}$, which Spivak leaves until the last chapters. Thus for the moment the proof will be taken on faith. In any case, it is better to build calculus upon abstract complete ordered fields than upon concrete real numbers.

5.4 Consequences of completeness

$\mathcal{N}$ is not bounded above

We've shown $\mathcal{N}$ has no upper bound in $\mathcal{N}$. Now we show $\mathcal{N}$ has no upper bound in $\mathcal{R}$.

Suppose for contradiction $\mathcal{N}$ has an upper bound. Since $\mathcal{N} \neq \emptyset$ then by completeness $\mathcal{N}$ has a least upper bound. Let $\alpha = \sup \mathcal{N}$. Then:

$$\begin{aligned} & \alpha \geq n \text{ for all } n \in \mathcal{N} \\ \implies & \alpha \geq n + 1 \text{ for all } n \in \mathcal{N} && \text{since } n + 1 \in \mathcal{N} \text{ if } n \in \mathcal{N} \\ \implies & \alpha - 1 \geq n \text{ for all } n \in \mathcal{N} \end{aligned}$$

Thus $\alpha - 1$ is *also* an upper bound for $\mathcal{N}$. This contradicts that $\alpha = \sup \mathcal{N}$. Therefore $\mathcal{N}$ is not bounded above, as desired.

$\sqrt{2}$ exists

We show $\sqrt{2} \in \mathcal{R}$. Let $S = \{y \in \mathcal{R} : y^2 < 2\}$. Obviously S is non-empty and has an upper bound. Thus by completeness property it has a least upper bound. Let $x = \sup S$. Note that $1 \in S$ and 2 is an upper bound of S. Thus $1 \leq x \leq 2$. We show $x^2 = 2$ by showing $x^2 \not< 2$ and $x^2 \not> 2$.

Case 1. Suppose for contradiction $x^2 < 2$. Let $0 < \epsilon < 1$ be a small number. Then

$$\begin{aligned}(x+\epsilon)^2 =& x^2 + 2\epsilon x + \epsilon^2 \\ &\leq x^2 + 4\epsilon + \epsilon && \text{since } x < 2 \text{ and } \epsilon < 1 \\ &= x^2 + 5\epsilon < 2 && \text{since } x^2 < 2 \text{ (by supposition), we can pick} \\ & && \text{a small enough } \epsilon \text{ to make this true}\end{aligned}$$

Thus there exists ϵ such that $(x+\epsilon)^2 < 2$. By definition of S it follows $x+\epsilon \in S$, which contradics that x is the least upper bound. Therefore $x^2 \nless 2$

Case 2. Suppose for contradiction $x^2 > 2$. Let $0 < \epsilon < 1$ be a small number. Then

$$\begin{aligned}(x-\epsilon)^2 =& x^2 - 2\epsilon x + \epsilon^2 \\ &\geq x^2 - 2\epsilon x && \text{since } \epsilon^2 > 0 \\ &\geq x^2 - 4\epsilon && \text{since } x \leq 2 \\ &> 2 && \text{since } x^2 > 2 \text{ (by supposition), we can pick} \\ & && \text{a small enough } \epsilon \text{ to make this true}\end{aligned}$$

Thus $(x-\epsilon)^2 > 2$, which by definition of S implies $x - \epsilon > y$ for all $y \in S$. So $x - \epsilon$ is an upper bound of S. We have a contradiction– since $x - \epsilon < x$, it follows x is not a least upper bound. Therefore $x^2 \ngtr 2$ as desired.

Since $x^2 \nless 2$ and $x^2 \ngtr 2$, it follows $x^2 = 2$ as desired.

Archimedean property

Handwavy definition: the Archimedean property states that you can fill the universe with tiny grains of sand.

Formal defition: let $\epsilon > 0$ be small and let $r > 0$ be large. Then there exists $n \in \mathcal{N}$ such that $n\epsilon > r$.

Proof: suppose for contradiction the property is false. Then there exist ϵ, r such that for all $n \in \mathcal{N}$, $n\epsilon \leq r$. Therefore $n \leq \frac{r}{\epsilon}$. This implies $\mathcal{N}$ is bounded, which a contradiction.

A useful special case is when $r = 1$. In this case the Archimedean property can be restated as follows. Let $\epsilon > 0$ be small. Then there exists $n \in \mathcal{N}$ such that $n\epsilon > 1$. Put differently, there exists $n \in \mathcal{N}$ such that $\frac{1}{n} < \epsilon$.

A few more notes on the Archimedean property:

- Obviously the Archimedean property follows from completeness, as shown above.

- The Archimedean property is true in $\mathcal{Q}$ and can be proven without being assumed[8].
- Completeness does not follow from the Archimedean property. The proof is easy: the Archimedean property holds on $\mathcal{Q}$, and we know $\mathcal{Q}$ is not complete as shown above.

Density

Let $x, y \in \mathcal{R}$. Then S is a **dense subset** of $\mathcal{R}$ if there is an element of S in (x, y). Put differently, there is an element of S between any two points in $\mathcal{R}$.

- Obviously $\mathcal{R}$ is a dense subset of itself (if $x, y \in \mathcal{R}$ then $\frac{x+y}{2} \in (x, y)$).
- Integers are not a dense subset of $\mathcal{R}$. E.g. there is no integer between 1.1 and 1.9.
- The set of positive numbers $\{x : x \in \mathcal{R}, x > 0\}$ is not a dense subset of $\mathcal{R}$. E.g. there is no positive number between -2 and -1.

Claim: the set of rational numbers $\mathcal{Q}$ is dense.
Proof: let $x, y \in \mathcal{R}$ be given. Suppose we can show there exists a rational in (x, y) for $0 \leq x < y$. Then:

- Given $x < y \leq 0$, there is a rational r in $(-y, -x)$. So $-r$ is in (x, y).
- Given $x < 0 < y$, there is a rational r in $(0, y)$. So r is of course also in (x, y).

Thus all we must do is prove there exists a rational in (x, y) for $0 \leq x < y$.

Let $0 \leq x < y$ be given. By the Archimedean property there exists $n \in \mathcal{N}$ such that $\frac{1}{n} < y - x$. Because (a) $\mathcal{N}$ is unbounded and (b) $\mathcal{N}$ is well-ordered, there exists the least integer $m \in \mathcal{N}$ such that $m \geq ny$.

First, observe that

$$m - 1 < ny \qquad \text{or } m \text{ wouldn't be the } \textit{least} \text{ integer } m \geq ny$$
$$\implies \frac{m-1}{n} < y$$

[8] Excluding the proof here, but it's fairly simple

Second, suppose for contradiction $\frac{m-1}{n} \leq x$. Then

$$\begin{aligned}
&\frac{m-1}{n} \leq x \\
&\implies \frac{m}{n} - \frac{1}{n} \leq x \\
&\implies -\frac{1}{n} \leq x - \frac{m}{n} \\
&\implies \frac{1}{n} \geq \frac{m}{n} - x \\
&\implies \frac{1}{n} \geq y - x && \text{recall } m \geq ny \text{, thus } \frac{m}{n} \geq y
\end{aligned}$$

This is a contradiction, thus $\frac{m-1}{n} > x$.

Therefore $\frac{m-1}{n} \in (x, y)$ as desired.

Claim: the set of irrational numbers $\mathcal{R} \setminus \mathcal{Q}$ is dense.
Proof: let $x, y \in \mathcal{R}$ be given. By density of the rationals there exists $r \in \mathcal{Q}$ such that $\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$. Multiplying each side by $\sqrt{2}$, we get $x < \sqrt{2}r < y$. We know $\sqrt{2}r$ is irrational. Thus there exists an irrational number between any two numbers in $\mathcal{R}$, and the set of irrationals $\mathcal{R} \setminus \mathcal{Q}$ is dense as desired.