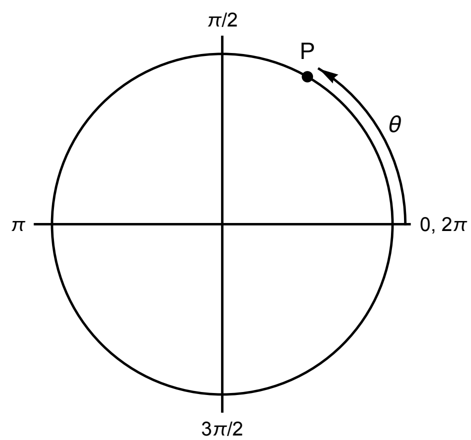
8 Trigonometric functions

Spivak covers variations of trigonometric functions as he goes through the book. I initially found it distracting– it's hard enough to understand $\epsilon - \delta$ limits; it's harder still if you're also trying to learn trig as you go. But trig is important, and sooner or later it's time to understand it. Because trig examples are crucial in understanding the chain rule in the next chapter, that time is now. Here I introduce the basics of trigonometric functions, and then cover everything trig-related from Spivak's early chapters that I ignored until now.

8.1 Definitions

Let P be a point on a unit circle $x^2+y^2=1$. Let θ be the length of the arc from $(1,0)$ to P, measured counterclockwise along the circle. Then the coordinates of P are $(\cos\theta, \sin\theta)$.[12]



The measure of angles by the length of the arc is in units called *radians*. Recall the circumference of a circle is $C = 2\pi r$, and so the circumference of a unit circle is 2π. Thus π represents a $180°$ angle. Some common angles in radians are $2\pi, \pi, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$, and $\frac{3\pi}{2}$. To convert these to degrees simply replace π with 180, and compute the fraction.

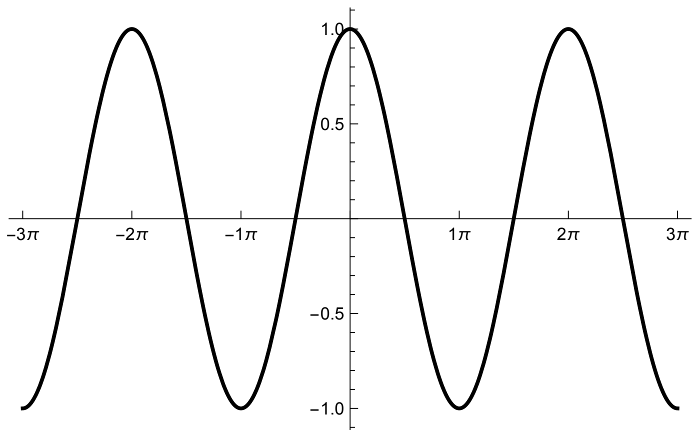
It should be self-evident that adding 2π to an angle results in the angle itself; and that adding $\frac{\pi}{2}$ to an angle shifts it by $90°$. Further:

$$(\cos 0, \sin 0) = (1,0) \qquad \text{and} \qquad (\cos\frac{\pi}{2}, \sin\frac{\pi}{2}) = (0,1)$$

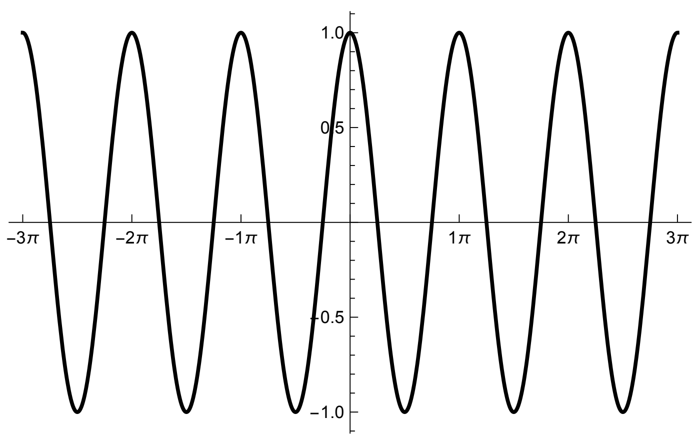
[12]The order is easy to remember– it's alphabetical.

8.2 Plotting

It is not too difficult to plot trigonometric functions. Consider some properties of cosine we've already seen (or can easily deduce): $\cos 0 = 1$, $\cos\frac{\pi}{2} = 0$, $\cos\pi = -1$. We've also seen that $\cos(x + 2\pi) = \cos x$. The x-axis below covers $[-3\pi, 3\pi]$ (i.e. a total length of 6π). Since cosine repeats every 2π, we should expect the graph to repeat thrice. And this is exactly what we see.



We can easily increase the frequency by plotting $y = \cos cx$. Here we double the frequency with $c = 2$:



8.3 Limits

Claim 1a: Let $f(x) = x \sin \frac{1}{x}$. Then $\lim_{x \to 0} f(x) = 0$.
Proof. Let $\epsilon > 0$ be given. We must show $0 < |x - 0| < \delta$ implies $|f(x) - 0| < \epsilon$. This is easy if you recall the co-domain of sin is $[-1, 1]$. This implies:

$$\left| x \sin \frac{1}{x} \right| \leq |x|$$

Thus fixing $|x| < \epsilon$ implies $\left| x \sin \frac{1}{x} \right| < \epsilon$ as desired.

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Claim 1b: Let $f(x) = x^2 \sin \frac{1}{x}$. Then $\lim_{x \to 0} f(x) = 0$.
Proof. By the same reasoning as above, $\left| x^2 \sin \frac{1}{x} \right| \leq x^2$. Thus fixing $|x| < \sqrt{\epsilon}$ implies $\left| x^2 \sin \frac{1}{x} \right| < \epsilon$ as desired.

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Claim 1c: Let $f(x) = \sqrt{|x|} \sin \frac{1}{x}$. Then $\lim_{x \to 0} f(x) = 0$.
Proof. By the same reasoning as above, $\left| \sqrt{|x|} \sin \frac{1}{x} \right| \leq \sqrt{|x|}$. Thus fixing $|x| < \epsilon^2$ implies $\left| \sqrt{|x|} \sin \frac{1}{x} \right| \leq \epsilon$ as desired.

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Claim 2: Let $f(x) = \sin \frac{1}{x}$. Then $\lim_{x \to 0} f(x)$ does not exist.
Proof. Let $\epsilon = \frac{1}{2}$. For any δ there exists $x = \frac{1}{\pi/2 + 2\pi \cdot n} < \delta$. Observe that $f(x) = 1 > \epsilon$. Thus $\lim_{x \to 0} f(x)$ does not exist as desired.

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Claim 3: Let $f(x) = \sin \frac{1}{x} = 0$. Then $\lim_{x \to \infty} f(x) = 0$.
Proof. Spivak states this without proof. The crux is that $|\sin t| < |t|$ for all real t (which we cannot prove until sin is defined later). But using this inequality the proof of the broader claim is simple.

Let $\epsilon > 0$ be given. We must show there exists N such that for all $x > N$, $|f(x)| < \epsilon$. Since $\left| \sin \frac{1}{x} \right| < \left| \frac{1}{x} \right|$, it suffices to ensure $\frac{1}{x} < \epsilon$, or $x > \frac{1}{\epsilon}$.

8.4 Continuity

Claim 1: $f(x) = \sin \frac{1}{x}$, $g(x) = x \sin \frac{1}{x}$ are not continuous at 0.
Proof. Neither function is defined at 0, and thus cannot be continuous at 0.

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Claim 2: Let

$$G(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then G is continuous at 0.

Proof. This is easy. We saw that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$, and $G(0) = 0$. Thus $\lim_{x \to 0} x \sin \frac{1}{x} = G(0)$, and thus G is continuous at 0.

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Claim 3: G is continuous at all a.
Proof. We already saw that G is continuous at 0 in claim 2. By using continuity theorems (and assuming sin is continuous), it's easy to see $x \sin \frac{1}{x}$ is continuous for all $x \neq 0$. Thus f is continuous for all a.

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Claim 4: Let

$$F(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ a, & x = 0 \end{cases}$$

Then F is *not* continuous at 0, for any choice of a.

Proof. We saw that $\lim_{x \to 0} x \sin \frac{1}{x}$ does not exist, thus F cannot be continuous at 0.